Chapter 3. Second order linear equations:

Section 3.1: Homogeneous equations w/ constant coefficients:

Recall that a second order ODE has the form

$$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt})$$

This equation is linear if f is of the form

$$f(t, y, \frac{dy}{dt}) = g(t) - p(t)\frac{dy}{dt} - q(t)\,y.$$

And the equation is precisely

$$y'' + p(t)\,y' + q(t)\,y = g(t).$$

Definition: A second order linear ODE is called homogeneous if the term $g(t)$ is zero for all t.

Otherwise, the equation is called non-homogeneous.

Remark: (1). Later on, ~~section~~ we will show that once a homogeneous equation $y'' + p(t)y' + q(t)y = 0$ can be solved, then $y'' + p(t)y' + q(t)y = g(t)$ (non-homogeneous equation) can also be solved. Thus homogeneous equations are more fundamental.

(2). In this chapter, we will concentrate on equations where $p(t)$, $q(t)$ are constants:

$$y'' + by' + cy = 0.$$

Example: $y'' - y = 0$.

The equation is saying: y is a function whose second derivative is the same as itself. A known solution is $y = e^t$, ~~act~~ And actually another function is $y = e^{-t}$. More generally $5 \cdot e^t$, $2e^{-t}$ ~~y=~~ are all solutions of the equation.

Observation: Suppose $\phi_1(t)$ and $\phi_2(t)$ are both functions of a 2nd order linear equation $y'' + P(t)\,y' + q(t)\,y = 0$. Then
homogeneous

$$C_1 \cdot \phi_1(t) + C_2 \cdot \phi_2(t) \text{ is also a solution, } C_1, C_2 \in \mathbb{R}.$$

A simple ~~con~~ fact is then

$$y = C_1 e^t + C_2 \cdot e^{-t}$$ are ~~also~~ all solutions of the equation $y'' - y = 0$.

Initial value problem: Suppose we want to find a solution of the equation $y'' - y = 0$ w/ initial condition $y(0) = 2$, $y'(0) = -1$. Then let $t = 0$ in $C_1 e^t + C_2 e^{-t}$, we get

$$C_1 + C_2 = y(0) = 2.$$

By differentiating ~~the eq~~ $y' = C_1 e^t - C_2 e^{-t}$.

$$\Rightarrow \; C_1 - C_2 = y'(0) = -1.$$

we find that $c_1 = \frac{1}{2}$, $c_2 = \frac{3}{2}$.

Hence $y = \frac{1}{2}e^{t} + \frac{3}{2}e^{-t}$.

We now turn to the more general ~~solution~~ equation

$$ay'' + by' + cy = 0.$$

We suppose that y is of the form $y = e^{rt}$, where r is a parameter to be determined. Then it follows that $y' = re^{rt}$, $y'' = r^2 e^{rt}$.

~~we~~ we then obtain

$$(ar^2 + br + c)\cdot e^{rt} = 0.$$

$$\Rightarrow ar^2 + br + c = 0.$$

This is called the characteristic equation for $a y'' + by' + cy = 0$.

Suppose r is a root of the polynomial equation, ~~then $y = e^{rt}$~~ there could be the following possibility:

1). ~~real roots~~ two different real ~~coefficients~~ roots.

2). real repeated root

3). complex roots.

We first consider case (1); say there are two roots $r_1 \neq r_2$. then $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are two solutions. And it follows that

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \text{ is also a solution.}$$

Now we want to find the particular solution satisfying initial conditions:

$$y(t_0) = y_0, \quad y'(t_0) = y_0'.$$

We have $c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} = y_0$.

$$c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} = y_0'.$$

It's not difficult to see that.

$$C_1 = \frac{y_0' - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0}, \quad C_2 = \frac{y_0 r_1 - y_0'}{r_1 - r_2} e^{-r_2 t_0}$$

($r_1 \neq r_2$ ~~to~~ is important).

Example: Find the solution of the initial value problem

$$y'' + 5y' + 6 = 0, \quad y(0) = 2, \; y'(0) = 3.$$

Solution: The characteristic equation is

$$r^2 + 5r + 6 = 0.$$

$$(r+2)(r+3) = 0 \implies r_1 = -2, \; r_2 = -3.$$

General solution $y = c_1 e^{-2t} + c_2 e^{-3t}$.

Initial conditions imply that.

$$\begin{cases} c_1 + c_2 = 2. \\ -2c_1 + (-3)c_2 = 3. \end{cases}$$

$$\begin{aligned} c_2 &= -7 \\ c_1 &= 9 \end{aligned} \implies y = 9e^{-2t} - 7e^{-3t}.$$

Section 3.2 Fundamental Solutions of linear Homogeneous equations:

Consider a ~~dif~~ 2nd order linear ODE

$$y'' + p y' + q y = 0. \qquad (*)$$

We can reformulate the differential equation in terms of differential operators.

We define a differential operator: say ~~is a~~ $y = y(t)$ is a function of t:

$$L[y] = y''(t) + p(t)\,y'(t) + q(t)\cdot y(t)$$

For instance: $p(t) = t^2$, $q(t) = 1+t$, $y(t) = \sin 3t$.

$$L[y](t) = (\sin 3t)'' + t^2(\sin 3t)' + (1+t)\sin 3t.$$

$$= -9\sin 3t + 3t^2\cos 3t + (1+t)\sin 3t.$$

~~Remark:~~ A function y is a solution of equation $(*)$ if and only if it is annihilated by the operator L.

Remark: The operator L is often written as $L = D^2 + p\cdot D + q$.

Theorem 3.2.1: Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0',$$

where p, q and g are continuous on an open interval I that containing the point t_0. Then there is a exactly one solution $y = \phi(t)$ of this problem, ~~the~~ and the solution exists throughout the interval I.

This theorem is saying the following:

1). Existence of ~~initial~~ solutions of initial value problem.

2). Uniqueness of solution.

3). The solution is defined throughout the interval.

Theorem 3.2.2: If y_1 and y_2 are two solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

, then linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of c_1 and c_2.

Question: If all ~~question~~ solutions of the equation $L[y] = y'' + p(t)y' + q(t) = 0$ are of the form $c_1y_1 + c_2y_2$?

We start from determining c_1 and c_2 from the initial conditions $y(t_0) = y_0$, $y'(t_0) = y_0'$. And we obtain the equations:

$$\cancel{c_1(t} \; c_1y_1(t_0) + c_2y_2(t_0) = y_0$$

$$c_1y_1'(t_0) + c_2y_2'(t_0) = y_0'.$$

This is a linear equation:

$$c_1 = \frac{y_0y_2'(t_0) - y_0'y_2(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}, \quad c_2 = \frac{-y_0y_1'(t_0) + y_0'y_1(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}$$

$$= \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} \quad ; \quad = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

We can see that in order for the ~~sol~~ expression to make sense, we need the denominators to be nonzero:

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \neq 0.$$

This is called the Wronskian determinant or simply the Wronskian of the solution y_1 and y_2.

We can conclude the following theorem:

<u>Theorem 3.2.3</u>: Suppose that y_1 and y_2 are two solutions of equation

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$ and that

the Wronskian $W = y_1 y_2' - y_2 y_1'$ is <u>non-zero</u> at the point t_0 where the initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0'$$ are assigned.

Then there is a choice of the constants c_1 and c_2 for which $y = c_1 y_1(t) + c_2 y_2(t)$ satisfies the initial value problem.

Example: We have seen that $y_1(t) = e^{-2t}$, $y_2(t) = e^{-3t}$ are solutions of the equation $y'' + 5y' + 6y = 0$.

The Wronskian of y_1 and y_2 is

$$W = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -e^{-5t}$$

So the Wronskian is nonzero for all values of t.

Theorem 3.2.4: If y_1 and y_2 are two solutions of the equation

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

and if there is $\underline{a}$ point t_0 where the Wronskian of y_1 and y_2 is nonzero, then the family of solutions

$$y = c_1 y_1(t) + c_2 y_2(t) \text{ w/ arbitrary coefficients}$$

c_1, c_2 includes every solution.

Proof: Let ϕ be any solution of the equation. To prove the theorem, we need to show $\exists\ c_1, c_2$, s.t.

$$\phi = c_1 y_1 + c_2 y_2.$$

Now let $y_0 = \phi(t_0)$, $y_0' = \phi'(t_0)$.

Then we consider the initial value problem

$$y'' + p(t)y' + q(t) = 0,\ y(t_0) = y_0,\ y'(t_0) = y_0'.$$

① On one hand, ϕ is certainly a solution. On the other hand, since $W(y_1, y_2)(t_0)$ is nonzero, we can find c_1, c_2 s.t.

$y = c_1 y_1(t) + c_2 y_2(t)$ is also a solution of the initial value problem. By the uniqueness of solutions.

$$\phi(t) = c_1 y_1(t) + c_2 y_2(t). \qquad \#$$

Section 3.3 Linear independence and the Wronskian.

Def: Two functions f and g are said to be linearly dependant on an interval I if there exists two constants k_1 and k_2, not both zero,

s.t. $k_1 f(t) + k_2 g(t) = 0$ for all t in I.

There is the following theorem relates linear independance (dependence) to the Wronskian.

Theorem 3.3.1: If f, g are differentiable functions on an open interval and if $W(f,g)(t_0) \neq 0$ for some $t_0 \in I$, then f and g are linearly independant on I. Moreover, if f, g are linealy dependence on I, then $W(f,g)(t) = 0$ for all $t \in I$.

Proof: Suppose $k_1 f(t) + k_2 g(t) = 0$.

$$\Rightarrow \quad k_1 f(t_0) + k_2 g(t_0) = 0$$

$$k_1 f'(t_0) + k_2 g'(t_0) = 0.$$

since $W = \begin{vmatrix} f(t_0) & g(t_0) \\ f'(t_0) & g'(t_0) \end{vmatrix} \neq 0 \Rightarrow \begin{array}{l} k_1 = 0 \\ k_2 = 0. \end{array}$

Definition: The expression $y = c_1 y_1(t) + c_2 y_2(t)$ w/ arbitrary coefficies is called the <u>general solution</u>.

The solutions y_1 and y_2 w/ a non-zero Wronskian are said to form a fundamental set of solutions of equation

$$y'' + P(t)\, y' + q(t)\, y = 0.$$

Example: Suppose $y_1(t) = e^{r_1 t}$, $y_2(t) = e^{r_2 t}$ are two solutions of a equation $y'' + by' + cy = 0$. ~~We show the~~ Suppose $r_1 \neq r_2$, then the Wronskian is

$$W = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1)\, e^{(r_1 + r_2)\cdot t} \neq 0$$

So y_1 and y_2 form a fundamental set of solutions.

Let us generalize this section:

To find the general solution of the differential equation

$$y'' + P(t)\, y' + q(t)\, y = 0, \quad \alpha < t < \beta$$

We must
1. Find two functions y_1 and y_2 that satisfy the equation in $\alpha < t < \beta$.

2. Make sure that there is a pt in the interval where the Wronskian W of y_1, y_2 is nonzero.

Then the general solution is $y = c_1 y_1(t) + c_2 y_2(t)$.

There is the following theorem, ~~descrit~~ giving an explicit expression of the Wronskian:

Theorem (Abel's Theorem): If y_1 and y_2 are solutions of the differential equation

$$L[y] = y'' + P(t)\, y' + q(t) \cdot y = 0, \text{ where}$$

p and q are continuous on an open interval I, then the Wronskian is given by

$$W(y_1, y_2)(t) = c \exp\left(-\int p(t)\, dt\right).$$

Thus the Wronskian is always zero or never zero.

Proof:

$$\begin{cases} y_1'' + P(t)\, y_1' + q(t)\, y_1 = 0. \\ y_2'' + P(t) \cdot y_2' + q(t) \cdot y_2 = 0. \end{cases}$$

$$-y_2\left(y_1'' + P(t)\, y_1' + q(t)\, y_1\right)$$

$$+ y_1\left(y_2'' + P(t)\, y_2' + q(t)\, y_2\right)$$

$$= \left(-y_2 y_1'' + y_1 y_2''\right) + P(t) \cdot \left(y_1 y_2' - y_1' y_2\right)$$

$$= W'(t) + P(t) \cdot W(t) = 0.$$

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$$W(t) = y_1 y_2' - y_2' y_1'$$

$$W'(t) = y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1''$$

$$= y_1 y_2'' - y_2 y_1''$$

Then there is the following theorem:

Theorem: Let y_1, y_2 be two solutions of the equation

$$y'' + p(t)y' + q(t)\cdot y = 0$$

then the following statements are equivalent:

1). The functions y_1, y_2 are a fundamental set of solutions

2). The functions y_1, y_2 are linearly independent on I.

3). $W(y_1, y_2)(t_0) \neq 0$ for some t_0

4) $W(y_1, y_2)(t) \neq 0$ for all t.

Section 3.4 Complex Roots ~~and~~ of the characteristic Equation:

We continue the discussion of the equation

$$ay'' + by' + cy = 0 \text{ , where } a, b, c \text{ are } \text{real numbers}$$

We were trying to find the solutions of the form $y = e^{rt}$, then r must be a root of the characteristic equation

$$ar^2 + br + c = 0.$$

Suppose now that the roots of the characteristic equation are conjugate complex numbers $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$.

Then the expressions for y are

$$y_1(t) = \exp[(\lambda + i\mu)t] \text{ , } y_2(t) = \exp[(\lambda - i\mu)t]$$